Monte Carlo Simulation of Hedonic Games

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ABSTRACT

Hedonic games have gained interest, by the academic community, in recent years because of their ability to model the grouping preferences of individuals. Hedonic games are an example of non-transferrable utility game in cooperative game theory. Cooperative game theory, or n-person game theory, is a branch of game theory that focuses on coalition formation within groups of players. Examples of hedonic games include the Marriage problem and Roommate problem. The literature on hedonic games has focused on finding analytical solutions because of the computational complexity of finding solutions for hedonic games. The problem with focusing on analytical results is that it does not give a sense of what the general properties of real solutions to hedonic games. In this study, we propose a Monte Carlo simulation to find the distributions of properties of hedonic game solutions. This simulation will involve the random generation a vast number of hedonic games and their solutions. The solutions are found using the core concepts from cooperative game theory. The form of a solution, to a hedonic game, is a coalition structure, which is a collection of disjoint coalitions that cover all the players. The property that we are interested in is the number of coalition structures in the core set. Finding the solution a hedonic game has been shown to be NP-Complete; to overcome this computational limitation, various efficiency improvements were used in finding the solutions including Individually Rational Coalition Lists (IRCL). This paper presents some initial results from this research.

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INTRODUCTION

This paper investigates the numerical properties of cooperative games. Cooperative game theory, or n-person game theory, is a branch of game theory that focuses on coalition formation within groups of players. We focus in on hedonic games, which is a form of cooperative game theory. Hedonic games are a form of cooperative games where each player has a preference relation over the set of coalitions (or groups) which they could be a member. The numerical analysis conducted in this research uses a Monte Carlo simulation approach. Our Monte Carlo simulation generates random hedonic games and then solves them (i.e., finds their core). Finding the core to a hedonic game is not trivial and is computationally intractable for large numbers of players. The outcomes from our numerical analysis are empirical distributions generated from 50,000 random games, for each game type. The game types are determined by the number of players considered.

The next section gives some background to hedonic games. This background is followed by a discussion on the algorithm used with the Monte Carlo simulations. Finally, the results and conclusions are given.

HEDONIC GAMES

A major division in game theory is the distinction between cooperative and non-cooperative games (McFadden, Tsai, Kadry, & Souba, 2012; Shubik, 2016). Cooperative game theory is gradually becoming more prevalent with its increased use in modeling cooperative relations in numerous fields: supply chain (Zhao, Wang, Cheng, Yang, & Huang, 2010), marketing (Ratchford, 2010), health care (McFadden et al., 2012) and maritime policy and management (Song & Panayides, 2002). An example of applying a cooperative game approach are situations where a group of decision-makers must decide whether to collaborate/undertake a project together to achieve their joint objectives and goals. An important concept in cooperative game theory is a coalition (C), which refers to the formation of sub-sets of players.

In literature, there are two main cooperative game theory models identified, one in which it is possible for players to transfer utility amongst themselves (the transferable utility games or TU games), and one where it is not possible (the non-transferable utility games or NTU games) (Airiau, 2010). In the scope of this work, we are particularly interested in a subclass of NTU game, e.g., Hedonic games. In hedonic games, one player cannot transfer any of the benefit (or utility) from being in a coalition to another member of a coalition. The outcome of the hedonic game is simply a coalition structure (CS). A coalition structure is a collection of disjoint coalitions that covers the set of players. To understand this phenomenon one can consider an example of a high school where the students represent the players and the clichés group represent the coalitions: a "geek" student might wish to be in the popular kids' cliché group but that does not mean that the popular kids would prefer their group to contain the geek. The non-transferable utility in this example will be if the popular kids did allow the geek in their group, then the geek may gain some social prestige but their social prestige cannot be transferred to any group member. Other popular examples of the hedonic game include the marriage and the roommate problems (Aziz & Savani, 2016).

Formal presentation of the hedonic games start with identifying and defining the coalition and the coalition structure, i.e., $\boldsymbol{C} \subseteq \boldsymbol{N},\ \boldsymbol{N} = \{\mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n}\}$ and $\boldsymbol{CS} = \{\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C_k}\}$ s.t. $\cup\, \mathbf{C_i} = \boldsymbol{N}\ \forall_{ij}\ \mathbf{C_i} \cap \mathbf{C_j} = \boldsymbol{\emptyset}.$ Here $\mathbf{N}$ is the set of the players. A hedonic game is given by a structure $\boldsymbol{G} = (\boldsymbol{N}, \succeq_{\mathbf{1}}, \dots, \succeq_{\mathbf{n}})$, where for all $\boldsymbol{i} \in \boldsymbol{N}$, the relation $\succeq_{\boldsymbol{i}} \subseteq \mathbf{N_i}\ \mathbf{x}\ \mathbf{N_i}$ is a complete, reflexive, and transitive preference relation over the possible coalitions of which i is a member. Where $\boldsymbol{N_i} = \{\boldsymbol{C} \cup \{\boldsymbol{i}\} | \boldsymbol{C} \subseteq \boldsymbol{N}\}$ is the collection of all subsets of $\boldsymbol{N}$ that contain $\boldsymbol{i}$, and $\boldsymbol{i} \in \boldsymbol{N}$.The interpretation of the structure $\boldsymbol{G}$ is that player $\boldsymbol{i}$ prefers to be in coalition $\boldsymbol{C_1}$ at least as much as $\boldsymbol{C_2}$, if we have following preference relation:

$\boldsymbol{C_1} \succeq_i \boldsymbol{C_2}$. A player will be indifferent to be in coalition $\boldsymbol{C_1}$ or $\boldsymbol{C_2}$ iff $\boldsymbol{C_1} \succeq_i \boldsymbol{C_2}$ and $\boldsymbol{C_2} \succeq_i \boldsymbol{C_1}$. The indifference relationship is denoted with $\sim_{\boldsymbol{i}}$.

The notion of the core is extended on the hedonic games adapted from Chalkiadakis, Elkind, & Wooldridge (2011). The core of a game is the set of coalition structures that are not dominated by any coalition. A dominated coalition structure is one where there exists a group of players which, if they formed a coalition, would prefer that newly formed coalition to the coalitions they are currently assigned. Thus, for a given member of the core, none of the players have an incentive to form a new coalition. In this research, it is the size of the core set that is of interest to us. Beyond the core set, there are various other solution concepts in hedonic games:

Core stability: A hedonic game is core stable if it has a non-empty core. Thus, if a game has a non-empty core, then there exists a coalition structure such that no set of players in the game would prefer to defect from the coalition structure and form a coalition of their own; that is, there exists a coalition structure that is not blocked. The notation for blocking is: $C \subseteq N$ blocks CS if $C \succ_i CS_i$ for all $i \in \mathrm{C}$

Individual rationality: If CS is individually rational, then every player does at least as well in CS as it would do alone ($CS_i \succeq_i \{i\}$ for all $i \in N$**)**

Nash stability: Nash stability indicates that no player would want to join other coalitions or form a singleton coalition, assuming no changes will occur with other coalitions. The formal notation for the Nash stability: $\forall_i \in N$**,** $CS_i \succeq_i C_{\mathrm{k}} \cup \{i\}$ for all $C_{\mathrm{k}} \in \mathrm{CS} \cup \{\emptyset\}$

Individual stability occurs when no player can move to another coalition that it prefers without making some members of that coalition unhappy. If there does not exist $i \in N$ and $C \in \mathrm{CS} \cup \{\emptyset\}$ such that $\mathrm{C} \cup \{i\} \succ_i CS_i$ and $\mathrm{C} \cup \{i\} \succ_j C$ for all $j \in C$

Contractually individual stability: A CS is contractually individually stable if there do not exist $\mathrm{i} \in N$ and $C \in \mathrm{CS} \cup \{\emptyset\}$ such that: (1) $\mathrm{C} \cup \{i\} \succ_i CS_i$ and $\mathrm{C} \cup \{i\} \succ_j C$ for all $j \in C$, and (2) $CS_i \setminus \{\mathrm{i}\} \succeq_{\mathrm{j}} CS_i \forall j \in CS_i$ {i}

Note that the solution concepts, i.e., core stability, individual rationality, individual stability, and contractual Nash stability, are interrelated (Chalkiadakis, Elkind, & Wooldridge, 2011). Individual stability implies contractually individual stability. Nash stability implies individual stability. However, the core stability does not imply Nash stability, nor does Nash stability imply core stability. Core stability does not imply individual stability. Below we define the stability concepts using CS for hedonic game $G = (N, \succeq_{\mathbf{1}}, \ldots, \succeq_n)$.

The next section of this work presents a way how to find the core of the hedonic game using Monte Carlo simulation.

ALGORITHM FOR FINDING THE CORE

There is no easy way to find the core of a game; in fact, it is NP-complete (Ballester, 2004). The brute force method (or naïve method) is to look at each coalition structure individually and then compare each possible coalition to see if it is dominated (Chalkiadakis et al., 2011). Considering that there are 2^n possible coalitions and the number of coalition structures is B^n, determined by Bell's number (which is exponential to the exponential in growth), solving a hedonic game become computationally intractable for relatively small numbers of agents; finding the core of game larger than 20 players is not feasible using current computing technology. There have been attempts to reduce that computational complexity, but these mainly involve restricting the form of the game to ensure it as nice properties (for example, induced subgraph games, see Deng and Papadimitriou (1994)). It is a variation of the naïve method that we used to solve our hedonic games. An overview of the algorithm that we used is given in Figure 1.

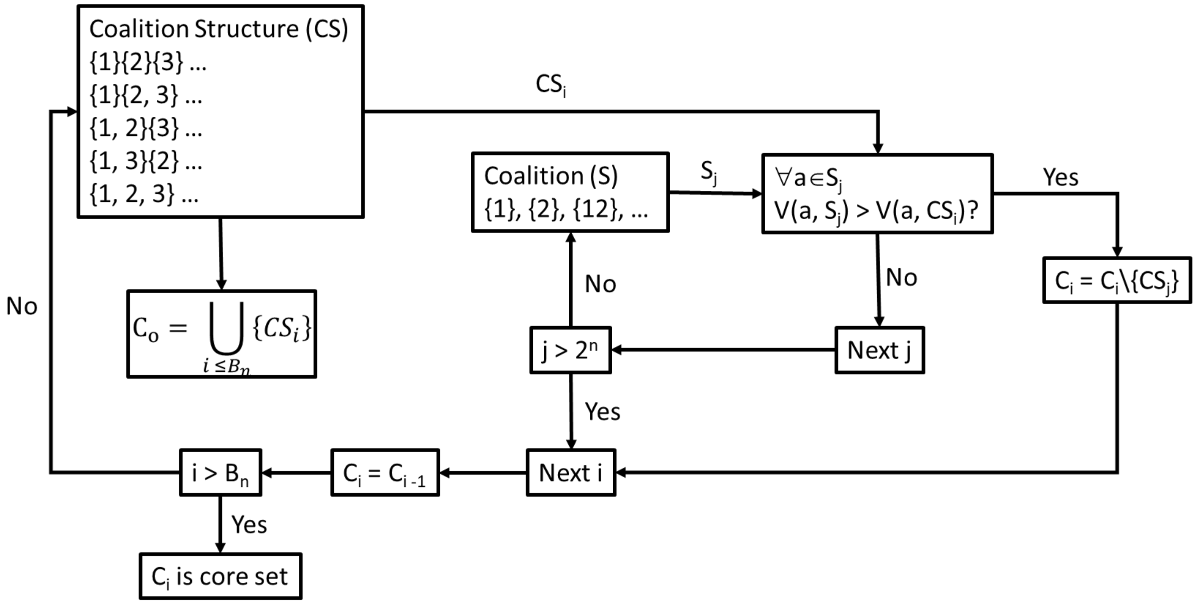


Figure 1. Flow diagram of our process for solving a random Hedonic game

The algorithm contains two loops. The outer loop considers each coalition structure, CS, in turn. The inner loop compares a particular coalition structure to every possible coalition, S. This comparison is made by comparing the preference of each player 'a' in S, represented by V(a, S), to the preference of that player in their current coalition, represent as V(a, CS); if all the members of S prefer S over their coalitions in CS, then CS is excluded from set C, which is initialized to contain all coalition structures. Once all the CS have been evaluated, then the set C represent the core.

The generation if the list of partition structures is not trivial and an effective algorithm was used to generate it (Djokić, Miyakawa, Sekiguchi, Semba, & Stojmenović, 1989). This algorithm uses the fact that any partition of players can be represented by a single integer, leading with a one. For example, 12123 represents a game of five players where players 1 and 3 are in a coalition, {1,3}; players 2 and 4 are in a coalition, {2, 4}, and player 5 is in its singleton coalition, {5}. This representation of a coalition structure ensures it is both disjoint and covers the set of players. The generation of the list of possible coalitions was done using a one-to-one correspondence between the coalitions and the binary numbers (e.g., 101 represents the coalition of players 1 and 3 in a three-player game, 100 represents the singleton coalition of player 1).

In a hedonic game, each player will rank each coalition that it is a number. To represent this, an integer was assigned to each coalition for each of its members. The number represents the players ranking of a particular subset, with a higher number representing a higher ranking. We purposefully did not allow a player to assign the same ranking to any two coalitions thus ensure a strict preference relation.

To speed up comparison, the algorithm had some minor adaptions. The main adaption was the inclusion of Individually Rational Coalition Lists (IRCL). IRCL removes all coalitions from the comparison step that are dominate by a singleton coalition. This reduces the number of coalition structure – coalition comparison that need to be conducted. More details of IRCL can be found in Ballester (2004)

METHOD

Our Monte Carlo experiment involves the repeated generation a random hedonic game and then determining its core set using the algorithm described above. The experiment was repeated for a different number of players in a game, ranging from three to seven. The results from games of one or two players can easily be solved analytically and have been included in the results for completeness.

The generation of a random hedonic game was done by randomly generating a preference relationship for each of the n players. For a given player, the numbers {1, 2, …., 2^{n-1}} where randomly allocated to each of the 2^{n-1} possible coalitions that that given player could be a member. This number represents the rank which the player gives that coalition. This process was done using the binary representation of each subset, and we have excluded the details of this process from this paper. The random number generator used in this process was the Mersenne Twister (Matsumoto & Nishimura, 1998).

For each game size (number of players), we generated 50,000 games and found the core for each. This limit was chosen because of the computational time required to calculate the core of a game of seven players, which was the upper limit of the game size chosen for this research. The number of coalition structures that must be checked is Bell's number, which is exponential to the exponential in growth; and each must be checked against an exponential number of possible coalitions; hence the number check growth excessively.

The focus of the results is on the size of the core, i.e., for a given game, how many coalitions structures are not dominated.

RESULTS

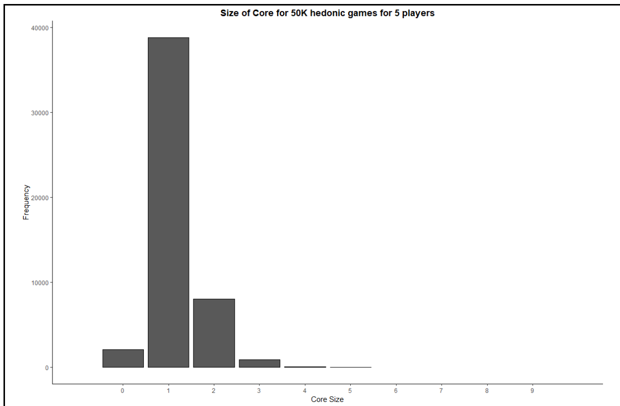
The size of the core was found for 250, 000 random hedonic games, which is summarized in Table 1. These results were games involving 3 to 7 players. The size of the core for games of one and two players can be solved analytically. For a game of only one player, there is only one possible coalition {1} and, thus, only one possible coalition structure. This coalition structure cannot be dominated, as no others exist, so it is in the core of the game. Hence, all hedonic games of one player have one coalition structure in their core. Finding the core of a two-player game is slightly more complex.

Consider the case when there are only two agents. Each agent could be exactly two coalitions: a singleton and the grand coalition. There are also exactly two possible coalition structure for this case: both agents are in their singleton coalition {1}{2}, and {1, 2}, the grand coalition. The core of hedonic games with two agents contains only one coalition structure; to understand this, we will consider all the possible game cases. The first case is when at least one agent prefers their singleton coalition (to the grand coalition) and the second case is when no agent prefers their singleton coalition. In the first case, the only element of the core is {1}{2} since at least one agent has no incentive to join (or remain) in the grand coalition. In this case, the singleton coalition structure is called Nash stable (Chalkiadakis et al., 2011). The grand coalition is unstable because at least one agent would prefer to be in their own singleton; this means that there is one and only one coalition structure in the core when at least one agent prefers their singleton coalition. Consider the second case, where neither player prefers their singleton coalition; this implies that both prefer the grand coalition (to their singleton sets) so only the coalition structure that is Nash stable is {1, 2}. Hence, in the case of two agents, there always exists a single member of the core.

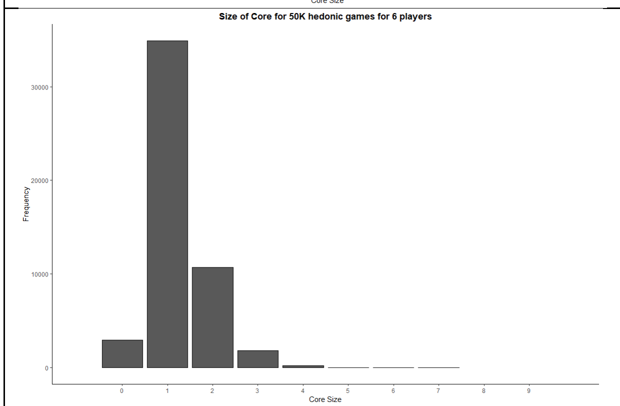
Table 1. Monte Carlo simulation results showing the results of the core size basis on 50,000 runs per game size

	Size of Core									
# Players	0	1	2	3	4	5	6	7	8	9
1	0	50,000*	0	0	0	0	0	0	0	0
2	0	50,000*	0	0	0	0	0	0	0	0
3	367	47,482	2,151	0	0	0	0	0	0	0
4	1,116	43,595	4,997	279	13	0	0	0	0	0
5	2,106	38,850	8,801	895	64	4	0	0	0	0
6	2,964	34,913	10,722	1,843	249	30	5	1	0	0
7	2,554	36,984	8,164	1,863	360	64	9	1	0	1
*Derived analytically										

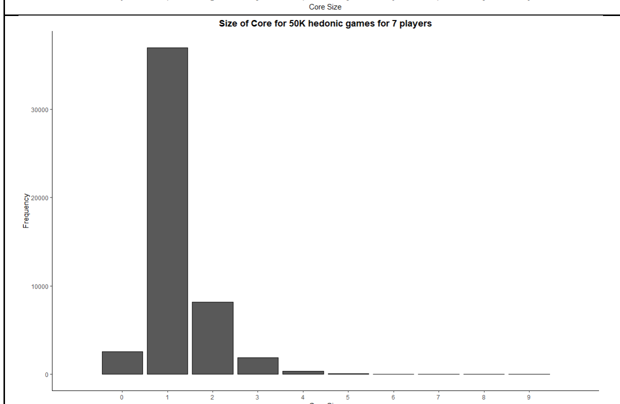
When we increase to three agents, the situation becomes more complex. For example, there are 13,824 possible games with three agents (though this number is reduce when consider the irrelevance of player labels). This complexity can lead to situations where the core is empty or there are multiple coalition structures in the core. An empty core can occur when there is a cycle of preferences among the players, i.e., player 1 prefers to be in a coalition in player 2 than player 3, player 2 prefers to be in a coalition with player 3 than player 1, and player 3 prefers to be in a coalition with player 1 than player 2. Having multiple coalitions structures in the core can occur, for example, consider a three player game were V(a, {a}) = 1, V(a, {a, b, c}) = 2, V(a, {a, b}) = 3, V(a, {a, c}) = 4, V(b, {b, c}) = 1, V(b, {b}) = 2, V(b, {a, b}) = 3, V(b, {a, b, c}) = 4, V(c, {a, c}) = 1, V(c, {b, c}) = 2, V(c, {c}) = 3, and V(c, {a, b, c}) = 4; in this game, there are two members of the core: {a, b, c} and {a, b}{c}. We leave it to the reader to check these coalition structures are not dominated.



(a) Five-players



(b) Six-players



(c) Seven-players

Figure 2. The distribution of core sizes for games of (a) five-players, (b) six-players, and (c) seven-players

The results indicate that as the number of players increases then so does the potential maximum size of the core. This makes sense because more players mean each coalition structure is larger which allows for more “distance” between coalition structures; by distance, it is meant the number of coalition changes that would have be made to transform one coalition structure into another. These distances make it possible to have multiple coalition structures in the core.

A core with a size of one is the most common result for all numbers of players. However, as the number of players increases, the percentage of games with a core size of one decreases. It would be tempting to conclude that an increase in numbers leads to a larger spread of core sizes (and the variances support this). However, notice that the number of results with a core size of one is larger in the games of seven-players than in the games of six-players; this suggests that the increase in spread might not be absolute. There does seem to be some similarity between the three largest games, which are shown in Figure 2.

One alternative possibility is that the distribution of core sizes is converging to a certain distribution (probably a lognormal distribution). To test this hypothesis, a one-way ANOVA test was conducted to see if the results for the three largest games could be concluded to be similar. The test results found that the distributions, for the different number of players, were significantly different (p-value < 0.0001). Hence, it was concluded that the distribution of core sizes has not converged. For future work, games with a larger number of agents will be considered to determine if the distributions are converging to a log-normal distribution.

Monte Carlo simulation is a random process so there will be random fluctuations in the results generated. This could explain the dip in the number of cores of size one for the six-player games. However, given the size of the change in the number of cores of size one (38K for five-players to 34K for six-players to 36K for seven-players); this seems unlikely. To investigate this phenomenon further, future work will include conducting a significantly larger number of runs to see if the dip is still present.

CONCLUSIONS

This paper presents the results of a Monte Carlo simulation to investigate the properties of hedonic games numerically. Hedonic games are a form of cooperative games where each player has a preference relation over the set of coalitions (or groups) which they could be a member. The numerical investigation in hedonic games was conducted by generating thousands of random hedonic games and finding how many solutions each game has, i.e., the size of the game's core. The results focused on the distribution of the core for different numbers of players in the game. It was found that that having a single core (solution) was the most common result for a game. Also, the distribution looked like they might be converging to some distribution through the statistical test indicated that this was not the case. Future work will include doing more runs for each size of the game and consider games with a larger number of players.

Common wisdom of cooperative games is the most games do not have a solution (the core is empty). Our analysis indicates the opposite is true and that the majority of hedonic games have a single solution. Having only a single solution is ideal because having multiple solutions means you have to choose one of them, which has caused problems for game theorists for decades (Harsanyi & Selten, 1988). By demonstrating this misconception about cooperative game theory, it is hoped that game theorist will be more willing to use cooperative game theory in the future.

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